TIMED AUTOMATA

LECTURE 13

TIMED AUTOMATA

Determinism

- DTA $\subset$ NTA
- Event-clock automata

Emptiness

- Region automaton
- Zone graph + simulations

Universality / Inclusion

- DTA : use the complement
- NTA : undecidable with ≥ 2 clocks

Extensions

Let $T\Sigma^*$ denote the set of **all timed words**

Universality: Given A, is $\mathcal{L}(A) = T\Sigma^*$?

Inclusion: Given A, B, is $\mathcal{L}(B) \subseteq \mathcal{L}(A)$?

Universality and inclusion are **undecidable** when A has **two clocks** or more

A theory of timed automata

Alur and Dill. *TCS'94*

A decidable case of the inclusion problem

Universality: Given A, is $\mathcal{L}(A) = T\Sigma^*$?

Inclusion: Given A, B, is $\mathcal{L}(B) \subseteq \mathcal{L}(A)$?

One-clock restriction

Universality and inclusion are **decidable** when A has at most **one clock**

On the language inclusion problem for timed automata: Closing a decidability gap

Ouaknine and Worrell. *LICS'05*

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One-clock restriction

Universality and inclusion are **decidable** when A has at most **one clock**

On the language inclusion problem for timed automata: Closing a decidability gap

Ouaknine and Worrell. *LICS'05*

In this lecture: universality for one clock TA

Step 0:

Well-quasi orders and Higman's Lemma

Quasi-order

Given a set $\mathcal{Q}$, a quasi-order is a **reflexive** and **transitive** relation:

$$\sqsubseteq \quad \subseteq \quad \mathcal{Q} \times \mathcal{Q}$$

- $(\mathbb{N}, \leq)$
- $(\mathbb{Z}, \leq)$

$$\text{Let } \Lambda = \{A, B, \ldots, Z\}, \quad \Lambda^* = \{\text{set of words}\}$$

- $(\Lambda^*,$ lexicographic order $\sqsubseteq_L)$: $AAAB \sqsubseteq_L AAB \sqsubseteq_L AB$
- $(\Lambda^*,$ prefix order $\subseteq_P)$: $AB \subseteq_P ABA \subseteq_P ABAA$
- $(\Lambda^*,$ subword order $\preccurlyeq)$ $HIGMAN \preccurlyeq HIGHMOUNTAIN$ [OW'05]

Well-quasi-order

An infinite sequence $\langle q_1, q_2, \ldots \rangle$ in $(Q, \sqsubseteq)$ is saturating if $\exists\, i < j : q_i \sqsubseteq q_j$

A quasi-order $\sqsubseteq$ is a well-quasi-order (wqo) if **every** infinite sequence is saturating

- $(\mathbb{N}, \leq)$: wqo
- ✗ $(\mathbb{Z}, \leq)$ $-1 \geq -2 \geq -3 \ldots \ldots$
- $(\Lambda^*,\ $ lexicographic order $\sqsubseteq_L)$:
- $(\Lambda^*,\ $ prefix order $\subseteq_P)$:
- $(\Lambda^*,\ $ subword order $\preccurlyeq)$

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- $(\mathbb{N}, \leq)$ ✓
- $(\mathbb{Z}, \leq)$ × $-1 \geq -2 \geq -3, \ldots$
- $(\Lambda^*,$ lexicographic order $\sqsubseteq_L)$: × $B \sqsupseteq_L AB \sqsupseteq_L AAB \ldots$
- $(\Lambda^*,$ prefix order $\subseteq_P)$: × $B,\ AB,\ AAB, \ldots$
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- $(\Lambda^*,$ subword order $\preccurlyeq)$?

Higman's lemma

Let $\sqsubseteq$ be a quasi-order on Λ

Define the induced **monotone domination order** $\preccurlyeq$ on Λ^* as follows:

$a_1 \ldots a_m \preccurlyeq b_1 \ldots b_n$ if there exists a **strictly increasing** function

$f : \{1, \ldots, m\} \mapsto \{1, \ldots, n\}$ s.t

$\forall\, 1 \leq i \leq m : \; a_i \sqsubseteq b_{f(i)}$

Higman's lemma

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$$\forall\, 1 \leq i \leq m : a_i \sqsubseteq b_{f(i)}$$

Higman'52

If $\sqsubseteq$ is a wqo on Λ, then the induced monotone domination order $\preccurlyeq$ is a wqo on Λ^*

Subword order

$$\Lambda \quad := \quad \{A, B, \ldots, Z\}$$
$$\sqsubseteq \quad := \quad x \sqsubseteq y \text{ if } x = y$$

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Induced monotone domination order $\preccurlyeq$ is the subword order

$$HIGMAN \preccurlyeq HIGHMOUNTAIN$$

By Higman's lemma, $\preccurlyeq$ is a wqo too

If we start writing an **infinite sequence** of words, we will **eventually** write down a **superword** of an earlier word in the sequence

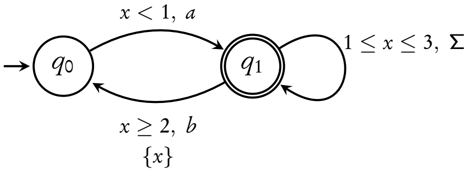
Step 1:

A naive procedure for universality of one-clock TA

Terminology

Let $A = (Q, \Sigma, Q_0, \{x\}, T, F)$ be a timed automaton with one clock

- **Location:** $q_0, q_1, \dots \in Q$
- **State:** (q, u) where $u \in \mathbb{R}_{\geq 0}$ gives value of the clock
- **Configuration:** **finite** set of states



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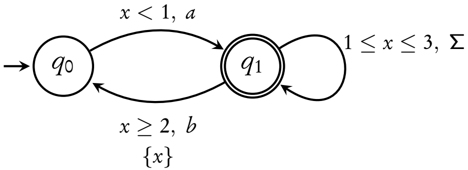
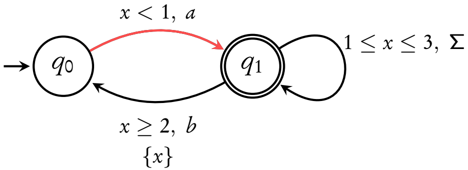
- **Location:** $q_0, q_1, \dots \in Q$
- **State:** (q, u) where $u \in \mathbb{R}_{\geq 0}$ gives value of the clock
- **Configuration:** **finite** set of states $\{(q_1, 2.3), (q_0, 0)\}$



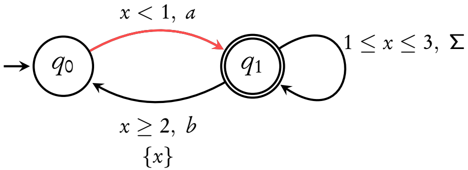
Transition between configurations:

$\{(q_0, 0)\} \xrightarrow{0.2,\ a}$



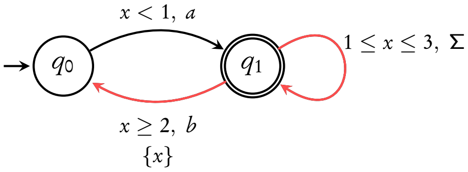
Transition between configurations:

$$\{(q_0, 0)\} \xrightarrow{0.2,\ a} \{(q_1, 0.2)\}$$



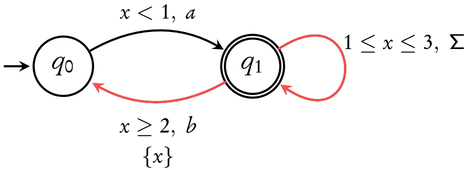
Transition between configurations:

$$\{(q_0, 0)\} \xrightarrow{0.2,\ a} \{(q_1, 0.2)\} \xrightarrow{2.1,\ b}$$



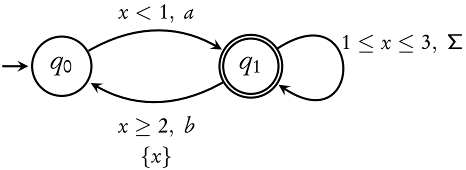
Transition between configurations:

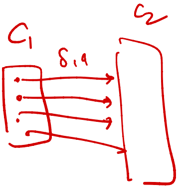
$$\{(q_0, 0)\} \xrightarrow{0.2,\ a} \{(q_1, 0.2)\} \xrightarrow{2.1,\ b} \{(q_1, 2.3), (q_0, 0)\} \ldots$$



Transition between configurations:

$$\{(q_0, 0)\} \xrightarrow{0.2,\ a} \{(q_1, 0.2)\} \xrightarrow{2.1,\ b} \{(q_1, 2.3), (q_0, 0)\} \ldots$$

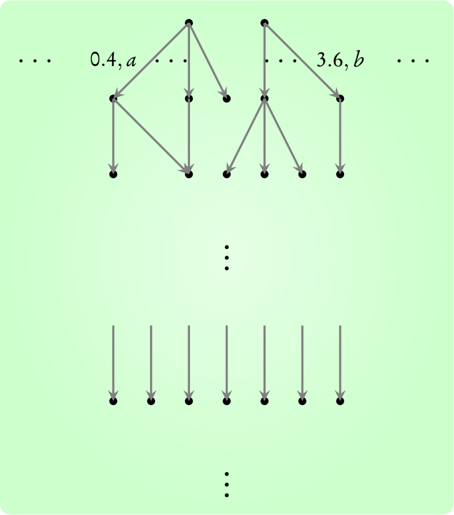




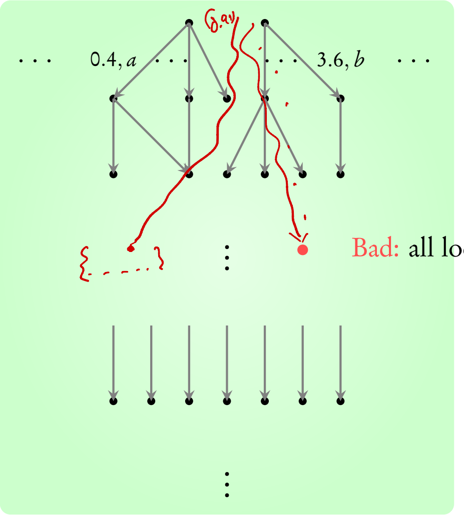
$$C_1 \xrightarrow{\delta,\ a} C_2 \text{ if}$$

$$C_2 = \{\ (q_2, u_2) \mid \exists (q_1, u_1) \in C_1 \text{ s. t. } (q_1, u_1) \xrightarrow{\delta,\ a} (q_2, u_2)\}$$

Labeled transition system of **configurations**



Labeled transition system of **configurations**

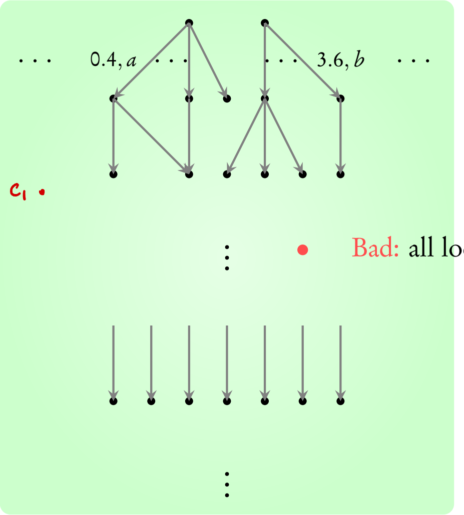


Bad: all locations **non-accepting**

$(\delta_1, a_1)(\delta_2, a_2) \ldots (\delta_n, a_n)$

- unique path for each word in this transition system.

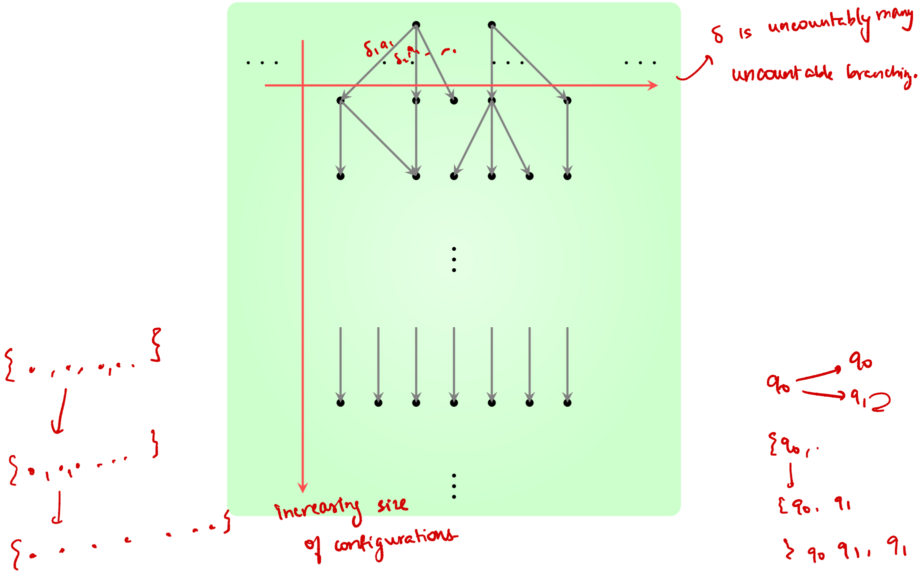
Labeled transition system of **configurations**



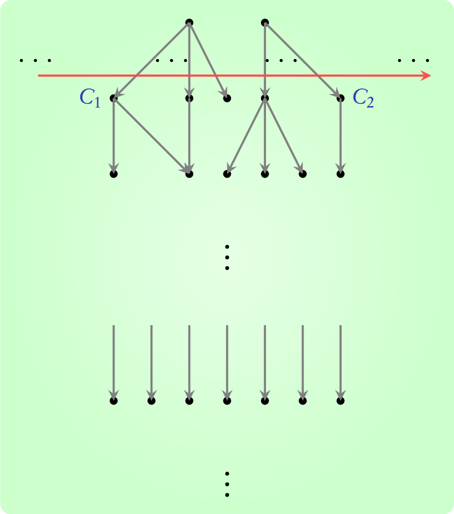
Bad: all locations **non-accepting**

Checking universality of A reduces to this question:

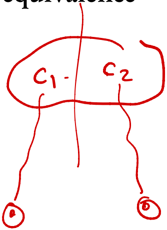
Is a bad configuration **reachable** from some **initial** configuration?



Need to handle **two dimensions** of infinity!



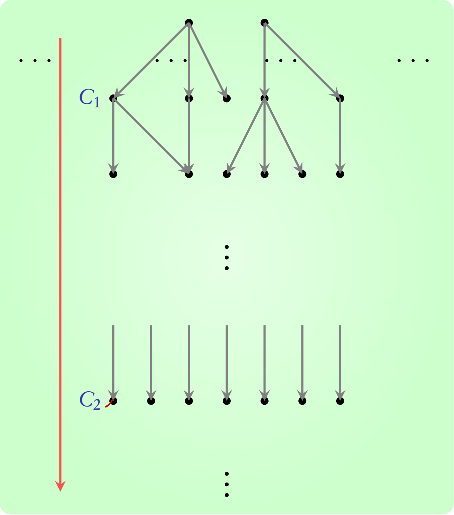
abstraction by **equivalence** $\sim$



$C_1 \sim C_2$ should imply:

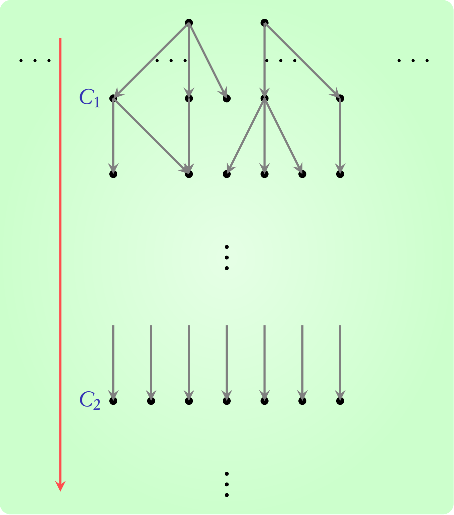
C_1 goes to a **bad** config. $\Leftrightarrow$ C_2 goes to a **bad** config.

finite **domination** order $\preccurlyeq$



$C_1 \preccurlyeq C_2$ should imply:

C_2 goes to a **bad** config $\Rightarrow$ C_1 goes to a **bad** config. too finite **domination** order $\preccurlyeq$



$C_1 \preccurlyeq C_2$ iff:

C_2 goes to a **bad** config $\Rightarrow$ C_1 goes to a **bad** config. too

No need to explore C_2!

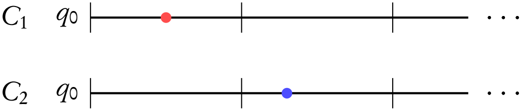
Step 2:

The equivalence

Credits: Examples in this part taken from one of **Ouaknine's talks**

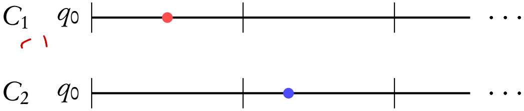
Equivalent configurations: Examples

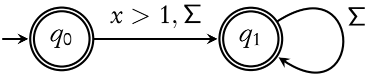
$$C_1 = \{(q_0, 0.5)\} \nsim C_2 = \{(q_0, 1.3)\}$$



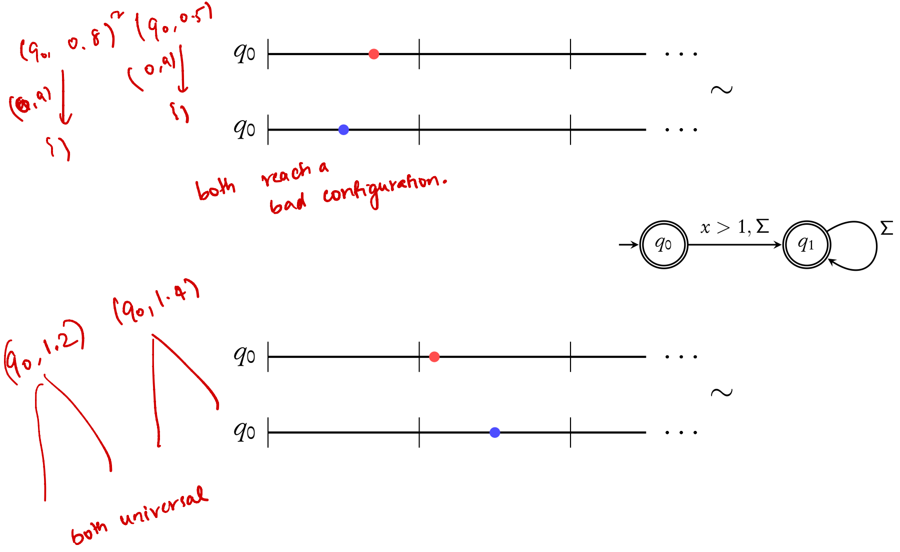
Equivalent configurations: Examples

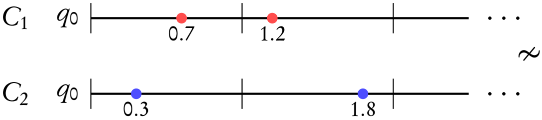
$$C_1 = \{(q_0, 0.5)\} \not\sim C_2 = \{(q_0, 1.3)\}$$





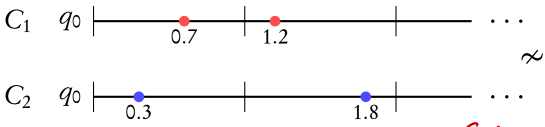
C_2 is universal, but C_1 **rejects** $(a, 0)$


(q0, 0.8)
(q0, 0.5)
{}
{}
q0
q0
~
both reach a bad configuration.
x > 1, Σ
Σ
q0
q1
(q0, 1.2)
(q0, 1.4)
q0
q0
~
both universal



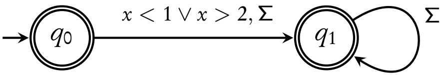
$C_1 = \{(q_0, 0.7), (q_0, 1.2)\}$

{(q0, 0.7), (q0, 1.2)}



{(q0, 0.3), (q0, 1.8)}

C3 = {(q0, 0.5), (q0, 1.4)}



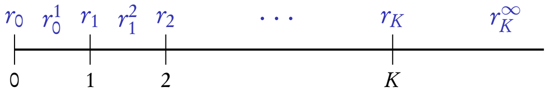
C1 ~ C3 ? ✓

C2 ~ C3 ? ✗

C_2 is universal, but C_1 **rejects** $(a, 0.5)$

Let K be the largest constant appearing in A

Define $REG = \{r_0, r_0^1, r_1, \ldots, r_K, r_K^\infty\}$



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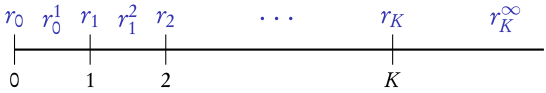

$$C = \{(q_1, 0.0), (q_1, 0.3), (q_1, 1.2), (q_2, 1.0), (q_3, 0.8), (q_3, 1.3)\}$$

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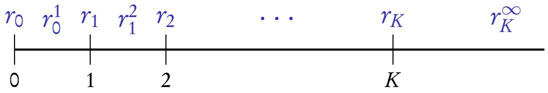

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$$\{(q_1, r_0, 0), (q_1, r_0^1, 0.3), (q_1, r_1^2, 0.2), (q_2, r_1, 0), (q_3, r_0^1, 0.8), (q_3, r_1^2, 0.3)\}$$

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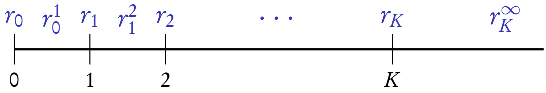

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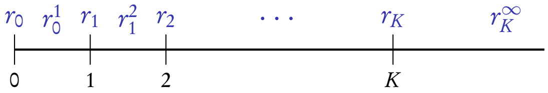
$$\{(q_1, r_0, 0), (q_2, r_1, 0)\}\{(q_1, r_1^2, 0.2)\}\{(q_1, r_0^1, 0.3)(q_3, r_1^2, 0.3)\}\{(q_3, r_0^1, 0.8)\}$$

Let K be the largest constant appearing in A

Define $REG = \{r_0, r_0^1, r_1, \ldots, r_K, r_K^\infty\}$



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$$\{(q_1, r_0, 0), (q_1, r_0^1, 0.3), (q_1, r_1^2, 0.2), (q_2, r_1, 0), (q_3, r_0^1, 0.8), (q_3, r_1^2, 0.3)\}$$

$$\{(q_1, r_0, 0), (q_2, r_1, 0)\} \; \{(q_1, r_1^2, 0.2)\} \; \{(q_1, r_0^1, 0.3)(q_3, r_1^2, 0.3)\} \; \{(q_3, r_0^1, 0.8)\}$$

$$H(C) = \{(q_1, r_0), (q_2, r_1)\} \; \{(q_1, r_1^2)\} \; \{(q_1, r_0^1)(q_3, r_1^2)\} \; \{(q_3, r_0^1)\}$$

Let K be the largest constant appearing in A

$$REG := \{r_0, r_0^1, r_1, \ldots, r_K, r_K^\infty\}$$

$$\Lambda := \mathcal{P}(\, Q \times REG \,)$$

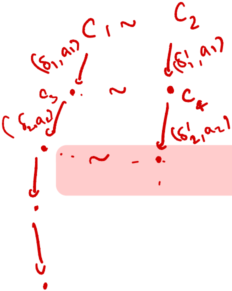
We can give $H : C \mapsto \Lambda^*$ that remembers:

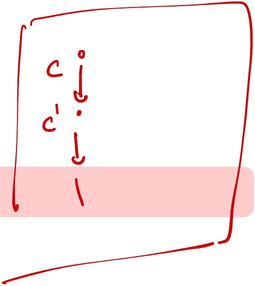
- **integral** part of the clock value (modulo K) in each state of C,
- **order** of **fractional** parts of the clock among different states in C

Equivalence

$$C_1 \sim C_2 \quad \text{if} \quad H(C_1) = H(C_2)$$

Equivalence

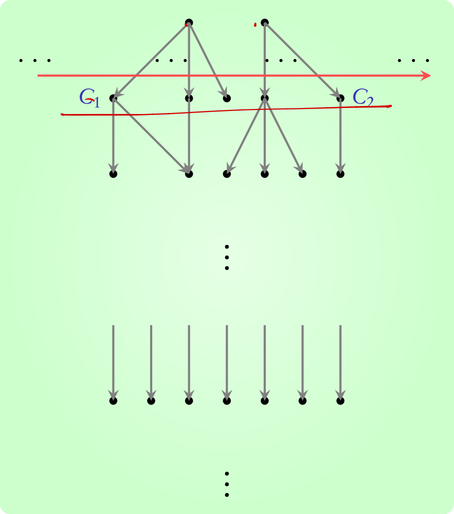




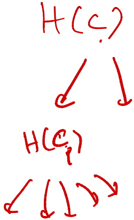
$C_1 \sim C_2$ if $H(C_1) = H(C_2)$

It can be shown that $\sim$ is a **bisimulation**

C_1 goes to a **bad** config. $\Leftrightarrow$ C_2 goes to a **bad** config.



abstraction by **equivalence** $\sim$



$C_1 \sim C_2$ iff:

C_1 goes to a **bad** config. $\Leftrightarrow$ C_2 goes to a **bad** config.